PYTHAGOREAN POWERS OF HYPERCUBES

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ABSTRACT. For $n \in \mathbb{N}$ consider the n-dimensional hypercube as equal to the vector space $\mathbb{F}_2^n$, where $\mathbb{F}_2$ is the field of size two. Endow $\mathbb{F}_2^n$ with the Hamming metric, i.e., with the metric induced by the ℓ_1^n norm when one identifies $\mathbb{F}_2^n$ with $\{0,1\}^n \subseteq \mathbb{R}^n$. Denote by $\ell_2^n(\mathbb{F}_2^n)$ the n-fold Pythagorean product of $\mathbb{F}_2^n$, i.e., the space of all $x = (x_1, \ldots, x_n) \in \prod_{j=1}^n \mathbb{F}_2^n$, equipped with the metric

$$\forall\, x, y \in \prod_{j=1}^n \mathbb{F}_2^n, \qquad d_{\ell_2^n(\mathbb{F}_2^n)}(x,y) \stackrel{\mathrm{def}}{=} \sqrt{\|x_1 - y_1\|_1^2 + \ldots + \|x_n - y_n\|_1^2}.$$

It is shown here that the bi-Lipschitz distortion of any embedding of $\ell_2^n(\mathbb{F}_2^n)$ into L_1 is at least a constant multiple of $\sqrt{n}$. This is achieved through the following new bi-Lipschitz invariant, which is a metric version of (a slight variant of) a linear inequality of Kwapień and Schütt (1989). Letting $\{e_{jk}\}_{j,k \in \{1,\ldots,n\}}$ denote the standard basis of the space of all n by n matrices $M_n(\mathbb{F}_2)$, say that a metric space (X, d_X) is a KS space if there exists $C = C(X) > 0$ such that for every $n \in 2\mathbb{N}$, every mapping $f : M_n(\mathbb{F}_2) \to X$ satisfies

$$\frac{1}{n}\sum_{j=1}^n \mathbb{E}\left[d_X\Big(f\Big(x + \sum_{k=1}^n e_{jk}\Big), f(x)\Big)\right] \leqslant C\mathbb{E}\left[d_X\Big(f\Big(x + \sum_{j=1}^n e_{jk_j}\Big), f(x)\Big)\right],$$

where the expectations above are with respect to $x \in M_n(\mathbb{F}_2)$ and $k = (k_1, \ldots, k_n) \in \{1, \ldots, n\}^n$ chosen uniformly at random. It is shown here that L_1 is a KS space (with $C = 2e^2/(e^2-1)$, which is best possible), implying the above nonembeddability statement. Links to the Ribe program are discussed, as well as related open problems.

1. Introduction

For a metric space (X, d_X) and $n \in \mathbb{N}$, the n-fold Pythagorean power of (X, d_X), denoted $\ell_2^n(X)$, is the space X^n, equipped with metric given by setting for every $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in X$,

$$d_{\ell_2^n(X)}\big((x_1, \ldots, x_n), (y_1, \ldots, y_n)\big) \stackrel{\mathrm{def}}{=} \sqrt{d_X(x_1, y_1)^2 + \ldots + d_X(x_n, y_n)^2}. \tag{1}$$

For $p \in [1, \infty]$, one analogously defines the ℓ_p powers of (X, d_X) by replacing in the right hand side of (1) the squares by p'th powers and the square root by the p'th root (with the obvious modification for $p = \infty$). When $(X, \|\cdot\|_X)$ is a Banach space and $p \in [1, \infty]$, one also commonly considers the Banach space $\ell_p(X)$ consisting of all the infinite sequences $x = (x_1, x_2, \ldots) \in X^{\aleph_0}$ such that $\|x\|_{\ell_p(X)}^p = \sum_{j=1}^\infty \|x_j\|_X^p < \infty$. One could give a similar definition of infinite ℓ_p powers for *pointed* metric spaces, but in the present article it will suffice to only consider n-fold powers of metric spaces for finite $n \in \mathbb{N}$.

Throughout the ensuing discussion we shall use standard notation and terminology from Banach space theory, as in [17]. In particular, for $p \in [1, \infty]$ and $n \in \mathbb{N}$, we use the notations $\ell_p = \ell_p(\mathbb{R})$ and $\ell_p^n = \ell_p^n(\mathbb{R})$, and the space L_p refers to the Lebesgue function space $L_p(0,1)$. We shall also use standard notation and terminology from the theory of metric embeddings, as in [18, 25]. In particular, a metric space (X, d_X) is said to admit a bi-Lipschitz embedding into a metric space (Y, d_Y) if there exists $s \in (0, \infty)$, $D \in [1, \infty)$ and a mapping $f : X \to Y$ such that

$$\forall\, x, y \in X, \qquad s d_X(x,y) \leqslant d_Y(f(x), f(y)) \leqslant D s d_X(x,y) \tag{2}$$

A. N. was supported by the NSF, the BSF, the Packard Foundation and the Simons Foundation.
G. S. was supported by the ISF and the BSF.

When this happens we say that (X,d_X) embeds into (Y,d_Y) with distortion at most D. We denote by $c_{(Y,d_Y)}(X,d_X)$ (or simply $c_Y(X), c_Y(X,d_X)$ if the metrics are clear from the context) the infimum over those $D\in[1,\infty]$ for which (X,d_X) embeds into (Y,d_Y) with distortion at most D. When $Y=L_p$ we use the shorter notation $c_{L_p}(X,d_X)=c_p(X,d_X)$.

A folklore theorem asserts that $\ell_2(\ell_1)$ is not isomorphic to a subspace of L_1. While this statement follows from a (nontrivial) gliding hump argument, we could not locate a reference to where it was first discovered; different proofs of certain stronger statements can be found in [14, Theorem 4.2], [27] and [28, Section 3]. More generally, $\ell_q(\ell_p)$ is not isomorphic to a subspace of L_1 whenever $q>p\geqslant 1$; the present work yields new information on this stronger statement as well, but for the sake of simplicity we shall focus for the time being only on the case of Pythagorean products.

Finite dimensional versions of the above results were discovered by Kwapień and Schütt, who proved in [16] that for every $n\in\mathbb{N}$, if $T:\ell_2^n(\ell_1^n)\to L_1$ is an injective linear mapping then necessarily $\|T\|\cdot\|T^{-1}\|\gtrsim\sqrt{n}$. Here, and in what follows, we use the convention that for $a,b\in[0,\infty)$ the notation $a\gtrsim b$ (respectively $a\lesssim b$) stands for $a\geqslant cb$ (respectively $a\leqslant cb$) for some universal constant $c\in(0,\infty)$. Below, the notation $a\asymp b$ stands for $(a\lesssim b)\wedge(b\lesssim a)$. By Cauchy–Schwarz, the identity mapping $Id:\ell_2^n(\ell_1^n)\to\ell_1^n(\ell_1^n)$ satisfies $\|Id\|\cdot\|Id^{-1}\|=\sqrt{n}$. So, the above lower bound of Kwapień and Schütt is asymptotically sharp as $n\to\infty$, up to the implicit universal constant.

By general principles, the above stated result of Kwapień and Schütt formally implies that

$$\lim_{n\to\infty} c_1(\ell_2^n(\mathbb{F}_2^n))=\infty, \tag{3}$$

where $\mathbb{F}_2^n$ is the n-dimensional discrete hypercube, endowed with the metric inherited from ℓ_1^n via the identification $\mathbb{F}_2^n=\{0,1\}^n\subseteq\mathbb{R}^n$. The deduction of (3) is as follows. Suppose for contradiction that $\sup_{n\in\mathbb{N}} c_1(\ell_2^n(\mathbb{F}_2^n))<\infty$. Since for every $\varepsilon>0$ every finite subset of ℓ_1 embeds with distortion $1+\varepsilon$ into $\mathbb{F}_2^m$ for some $m\in\mathbb{N}$ (see [7]), it follows from our contrapositive assumption that there exists $K\in[1,\infty)$ such that for every finite subset $X\subseteq\ell_1$ and every $n\in\mathbb{N}$ we have $c_1(\ell_2^n(X))\leqslant K$. By a standard ultrapower argument (as in [10]) this implies that $\sup_{n\in\mathbb{N}} c_1(\ell_2^n(\ell_1^n))\leqslant K$. Next, by using a w^*-Gâteaux differentiation argument combined with the fact that L_1^{**} is an $L_1(\mu)$ space (see [11] or [4, Chapter 7])) it follows that there exists a linear operator $T:\ell_2^n(\ell_1^n)\to L_1$ with $\|T\|\cdot\|T^{-1}\|\lesssim 2K$, contradicting the lower bound of Kwapień and Schütt. This proof of (3) does not yield information on the rate at which $c_1(\ell_2^n(\mathbb{F}_2^n))$ tends to ∞, a problem that we resolve here.

Theorem 1.1. *We have $c_1(\ell_2^n(\mathbb{F}_2^n))\asymp\sqrt{n}$.*

Note that if we write $Y\stackrel{\mathrm{def}}{=}\ell_2^n(\mathbb{F}_2^n)$ then $|Y|=2^{n^2}$, and therefore by Theorem 1.1 we have

$$c_1(Y)\asymp\sqrt[4]{\log|Y|}. \tag{4}$$

In light of (4), the following interesting open question asks whether or not $\ell_2^n(\mathbb{F}_2^n)$ is the finite subset of $\ell_2(\ell_1)$ that is asymptotically the furthest from a subset of L_1 in terms of its cardinality.

Question 1.2. *Suppose that $S\subseteq\ell_2(\ell_1)$ is finite. Is it true that $c_1(S)\lesssim\sqrt[4]{\log|S|}$?*

The following lemma (proved in Section 4 below) is a simple consequence of [2]. It shows that the answer to Question 1.2 is positive (up to lower order factors) for finite subsets $S\subseteq\ell_2(\ell_1)$ that are product sets, i.e., those sets of the form $S=X_1\times\ldots\times X_n\subseteq\ell_2^n(\ell_1)$ for some $n\in\mathbb{N}$ and finite subsets $X_1,\ldots,X_n\subseteq\ell_1$. This assertion is of course very far from a full resolution of Question 1.2. We conjecture that the answer to Question 1.2 is positive, and it would be worthwhile to investigate whether or not variants of the methods used in the proof of the main result of [2] are relevant here.

Lemma 1.3. *Suppose that $n\in\mathbb{N}$ and $X_1,\ldots,X_n\subseteq\ell_1$ are finite. Write $S=X_1\times\ldots\times X_n\subseteq\ell_2^n(\ell_1)$. Then*

$$c_1(S)\lesssim\sqrt[4]{\log|S|}\cdot\sqrt{\log\log|S|}=(\log|S|)^{\frac14+o(1)}.$$

1.1. **Metric Kwapień–Schütt inequalities.** In [16] (see also [15]) Kwapień and Schütt (implicitly) proved the following inequality, which holds for every $n\in\mathbb{N}$ and every $\{z_{jk}\}_{j,k\in\{1,\ldots,n\}}\subseteq L_1$.

$$\frac{1}{n}\sum_{j=1}^{n}\sum_{\varepsilon\in\{-1,1\}^n}\Big\|\sum_{k=1}^{n}\varepsilon_k z_{jk}\Big\|_1\lesssim\frac{1}{n!}\sum_{\pi\in S_n}\sum_{\varepsilon\in\{-1,1\}^n}\Big\|\sum_{j=1}^{n}\varepsilon_j z_{j\pi(j)}\Big\|_1, \tag{5}$$

where S_n denotes as usual the group of all permutations of $\{1,\ldots,n\}$.

The validity of (5) immediately implies the previously mentioned lower bound on the distortion of any linear embedding $T:\ell_2^n(\ell_1^n)\to L_1$. To see this, identify from now on $\ell_2^n(\ell_1^n)$ with $M_n(\mathbb{R})$ by considering for every $x=(x_1,\ldots,x_n)\in\ell_2^n(\ell_1^n)$ the matrix whose j'th row is $x_j\in\mathbb{R}^n$. With this convention, apply (5) to $z_{jk}=T(e_{jk})$, where e_{jk} is the n by n matrix whose (j,k) entry equals 1 and the rest of its entries vanish. Then for every $\varepsilon\in\{-1,1\}^n$ and $j\in\{1,\ldots,n\}$ we have

$$\Big\|\sum_{k=1}^{n}\varepsilon_k z_{jk}\Big\|_1\geqslant\frac{\left\|\sum_{k=1}^{n}\varepsilon_k e_{jk}\right\|_{\ell_2^n(\ell_1^n)}}{\|T^{-1}\|}=\frac{n}{\|T^{-1}\|}, \tag{6}$$

and for every $\pi\in S_n$ and $\varepsilon\in\{-1,1\}^n$ we have

$$\Big\|\sum_{j=1}^{n}\varepsilon_j z_{j\pi(j)}\Big\|_1\leqslant\|T\|\Big\|\sum_{j=1}^{n}\varepsilon_j e_{j\pi(j)}\Big\|_{\ell_2^n(\ell_1^n)}=\|T\|\sqrt{n}. \tag{7}$$

The only way for (6) and (7) to be compatible with (5) is if $\|T\|\cdot\|T^{-1}\|\gtrsim\sqrt{n}$.

In light of the above argument, it is natural to ask which Banach spaces satisfy (5), i.e., to obtain an understanding of those Banach spaces $(Z,\|\cdot\|_Z)$ for which there exists $K=K(Z)\in(0,\infty)$ such that for every $n\in\mathbb{N}$ and every $\{z_{jk}\}_{j,k\in\{1,\ldots,n\}}\subseteq Z$ we have

$$\frac{1}{n}\sum_{j=1}^{n}\sum_{\varepsilon\in\{-1,1\}^n}\Big\|\sum_{k=1}^{n}\varepsilon_k z_{jk}\Big\|_Z\leqslant\frac{K}{n!}\sum_{\pi\in S_n}\sum_{\varepsilon\in\{-1,1\}^n}\Big\|\sum_{j=1}^{n}\varepsilon_j z_{j\pi(j)}\Big\|_Z. \tag{8}$$

This question requires further investigation and obtaining a satisfactory characterization seems to be challenging. In particular, it seems to be unknown whether or not the Schatten trace class S_1 satisfies (8). Regardless, it is clear that the requirement (8) is a local linear property, and therefore by Ribe's rigidity theorem [29] it is preserved under uniform homeomorphisms of Banach spaces. In accordance with the Ribe program (see [3, 24]) one should ask for a bi-Lipschitz invariant of metric spaces that, when restricted to the class of Banach spaces, is equivalent to (8).

Following the methodology that was introduced by Enflo [8] (see also [9, 5]), a first attempt to obtain a bi-Lipschitz invariant that is (hopefully) equivalent to (8) is as follows. Consider those metric spaces (X,d_X) for which there exists $K=K(X)\in(0,\infty)$ such that for every $n\in\mathbb{N}$ and every $f:M_n(\mathbb{F}_2)\to X$ we have

$$\frac{1}{n}\sum_{j=1}^{n}\sum_{x\in M_n(\mathbb{F}_2)}d_X\Big(f\Big(x+\sum_{k=1}^{n}e_{jk}\Big),f(x)\Big)\leqslant\frac{K}{n!}\sum_{\pi\in S_n}\sum_{x\in M_n(\mathbb{F}_2)}d_X\Big(f\Big(x+\sum_{j=1}^{n}e_{j\pi(j)}\Big),f(x)\Big). \tag{9}$$

If X is in addition a Banach space and $\{z_{jk}\}_{j,k\in\{1,\ldots,n\}}\subseteq X$ then for $f(x)=\sum_{j=1}^{n}\sum_{k=1}^{n}(-1)^{x_{jk}}z_{jk}$ the inequality (9) becomes (8). However, for every integer $n\geqslant 3$ no metric space that contains at least two points can satisfy (9) with $K<n/2$, as explained in Remark 2.4 below. Thus, obtaining a metric characterization of the linear property (8) remains open.

We shall overcome this difficulty by first modifying the linear definition (8) so that it still implies the same nonembeddability result for $\ell_2^n(\ell_1^n)$, and at the same time we can prove that the reasoning that led to the metric inequality (9) now leads to an analogous inequality which does hold true for nontrivial metric spaces (specifically, we shall prove that it holds true for L_1).

Definition 1.4 (Linear KS space)**.** Say that a Banach space $(Z,\|\cdot\|_Z)$ is a linear KS space if there exists $C=C(X)\in(0,\infty)$ such that for every $n\in\mathbb{N}$ and every $\{z_{jk}\}_{j,k\in\{1,\ldots,n\}}\subseteq Z$ we have

$$\frac{1}{n}\sum_{j=1}^{n}\sum_{\varepsilon\in\{-1,1\}^n}\Big\|\sum_{k=1}^{n}\varepsilon_k z_{jk}\Big\|_Z\leqslant\frac{C}{n^n}\sum_{k\in\{1,\ldots,n\}^n}\sum_{\varepsilon\in\{-1,1\}^n}\Big\|\sum_{j=1}^{n}\varepsilon_j z_{jk_j}\Big\|_Z. \tag{10}$$

The difference between (10) and (8) is that we replace the averaging over all permutations $\pi\in S_n$ by averaging over all mappings $\pi:\{1,\ldots,n\}\to\{1,\ldots,n\}$. We shall see below that L_1 is a linear KS space. The same reasoning that leads to (9) now leads us to consider the following new bi-Lipschitz invariant for metric spaces.

Definition 1.5 (KS metric space)**.** Say that a metric space (X,d_X) is a KS space if there exists $C=C(X)\in(0,\infty)$ such that for every $n\in 2\mathbb{N}$ and every $f:M_n(\mathbb{F}_2)\to X$ we have

$$\frac{1}{n}\sum_{j=1}^{n}\sum_{x\in M_n(\mathbb{F}_2)}d_X\Big(f\Big(x+\sum_{k=1}^{n}e_{jk}\Big),f(x)\Big)\leqslant\frac{C}{n^n}\sum_{k\in\{1,\ldots,n\}^n}\sum_{x\in M_n(\mathbb{F}_2)}d_X\Big(f\Big(x+\sum_{j=1}^{n}e_{jk_j}\Big),f(x)\Big). \tag{11}$$

Remark 1.6. The reason why in Definition 1.5 we require (11) to hold true only when $n\in\mathbb{N}$ is even is that no non-singleton metric space (X,d_X) satisfies (11) when $n\geqslant 3$ is an odd integer. Indeed, suppose that $a,b\in X$ are distinct and that $n\geqslant 3$ is an odd integer. For $x\in M_n(\mathbb{F}_2)$ write $\sigma(x)=\sum_{j=1}^{n-1}\sum_{k=1}^{n}x_{jk}\in\mathbb{F}_2$. Define $f:M_n(\mathbb{F}_2)\to X$ by setting $f(x)=a$ if $\sigma(x)=0$ and $f(x)=b$ if $\sigma(x)=1$. For every $j\in\{1,\ldots,n-1\}$ and $x\in M_n(\mathbb{F}_2)$ we have $\sigma(x+\sum_{k=1}^{n}e_{jk})=\sigma(x)+n\neq\sigma(x)$, since n is odd. Consequently the left hand side of (11) is nonzero (since a and b are distinct). But, for every $x\in M_n(\mathbb{F}_2)$ and $k\in\{1,\ldots,n\}^n$ we have $\sigma(x+\sum_{j=1}^{n}e_{jk_j})=\sigma(x)+n-1=\sigma(x)$, since n is odd. Consequently the right hand side of (11) vanishes. This parity issue is of minor significance: in Remark 2.3 below we describe an inequality that is slightly more complicated than (11) but make sense for every $n\in\mathbb{N}$ and in any metric space, and we show that it holds true for L_1-valued functions. This variant has the same nonembeddability consequences as (11), albeit yielding distortion lower bounds that are weaker by a constant factor.

The following theorem is the main result of the present article. We shall soon see, in Section 1.2 below, how it quickly implies Theorem 1.1 (and more).

Theorem 1.7. *L_1 is a KS space. Namely, for all $n\in 2\mathbb{N}$ and every $f:M_n(\mathbb{F}_2)\to L_1$ we have*

$$\sum_{j=1}^{n}\sum_{x\in M_n(\mathbb{F}_2)}\Big\|f\Big(x+\sum_{k=1}^{n}e_{jk}\Big)-f(x)\Big\|_1\leqslant\frac{2n}{n^n-(n-2)^n}\sum_{k\in\{1,\ldots,n\}^n}\sum_{x\in M_n(\mathbb{F}_2)}\Big\|f\Big(x+\sum_{j=1}^{n}e_{jk_j}\Big)-f(x)\Big\|_1.$$

For every fixed $n\in 2\mathbb{N}$, the factor $2n/(n^n-(n-2)^n)$ above cannot be improved. So, L_1 satisfies (11) *for every $n\in 2\mathbb{N}$ with*

$$C=\sup_{n\in 2\mathbb{N}}\frac{2}{1-\left(1-\frac{2}{n}\right)^n}=\frac{2e^2}{e^2-1},$$

and this value of C cannot be improved.

Note that if $(Z,\|\cdot\|_Z)$ is a Banach space and $\{z_{jk}\}_{j,k\in\{1,\ldots,n\}}\subseteq Z$ then by considering the mapping $f:M_n(\mathbb{F}_2)\to Z$ given by $f(x)=\sum_{j=1}^{n}\sum_{k=1}^{n}(-1)^{x_{jk}}z_{jk}$ we see that if Z is a KS space as a metric space then it is also a linear KS space (with the same constant C). We do not know whether or not the converse holds true, i.e., we ask the following interesting open question: if a Banach space $(Z,\|\cdot\|_Z)$ is a linear KS space then is it also a KS space as a metric space? Understanding which Banach spaces are linear KS spaces is a wide-open research direction. In particular, we ask whether the Schatten trace class S_1 is a KS space as a metric space, or even whether it is a linear KS space.

There are inherent conceptual difficulties that indicate that our proof of Theorem 1.7 cannot be extended to the case of S_1 without a substantial new idea; see Remark 3.3 below.

Our proof of Theorem 1.7 consists of simple Fourier analysis combined with a nonlinear transformation; see Section 2 below. The simplicity of this proof indicates one of the values of generalizing linear inequalities such as (10) to their stronger nonlinear counterparts, since this brings genuinely nonlinear tools into play. In particular, we thus obtain a very simple proof of the linear inequality (10) through an argument which would have probably not been found without the need to generalize (10) to a metric inequality as part of the Ribe program.

1.2. **Embeddings of $\ell_q(\mathbb{F}_2^n,\|\cdot\|_p)$ into L_1.** Suppose that $q>p\geqslant 1$. By arguing as in (6) and (7) one deduces from the Kwapień–Schütt inequality (5) that for every $n\in\mathbb{N}$, every injective linear mapping $T:\ell_q^n(\ell_p^n)\to L_1$ must satisfy

$$\|T\|\cdot\|T^{-1}\|\gtrsim n^{\frac1p-\frac1q}. \tag{12}$$

Note that this conclusion was stated by Kwapień and Schütt in [16, Crollary 3.4] under the additional assumption that $q\leqslant 2$, but this restriction is not necessary. By a differentiation argument (see [4, Chapter 7]), it follows from (12) that

$$c_1\left(\ell_q^n(\ell_p^n)\right)\gtrsim n^{\frac1p-\frac1q}. \tag{13}$$

We previously deduced from the case $q=2$ and $p=1$ of (13) that $\lim_{n\to\infty}c_1(\ell_2^n(\mathbb{F}_2^n))=\infty$. This was done in the paragraph that followed (3), relying on the fact that any finite subset of ℓ_1 admits an embedding with $O(1)$ distortion into $\mathbb{F}_2^m$ for some $m\in\mathbb{N}$. The analogous assertion is not true for $p>1$, and therefore despite the validity of (13) it was previously unknown whether or not $\sup_{n\in\mathbb{N}}c_1(\ell_q(\mathbb{F}_2^n,\|\cdot\|_p))=\infty$. Our metric KS inequality of Theorem 1.7 answers this question.

Theorem 1.8. *Suppose that $1\leqslant p<q$ and $n\in\mathbb{N}$ then*

$$c_1\left(\ell_q^n\left(\mathbb{F}_2^n,\|\cdot\|_p\right)\right)\asymp n^{\frac1p-\frac1q}. \tag{14}$$

It is worthwhile to note here that while Theorem 1.8 yields a sharp asymptotic evaluation of $c_1\left(\ell_q^n\left(\mathbb{F}_2^n,\|\cdot\|_p\right)\right)$, the corresponding bound (13) in the continuous setting is not always sharp. Specifically, in Section 4 we explain that

$$c_1\left(\ell_q^n(\ell_p^n)\right)\asymp\begin{cases} n^{\frac1p-\frac1q} & \text{if } 1\leqslant p<q \text{ and } p\leqslant 2,\\ n^{1-\frac1p-\frac1q} & \text{if } 2\leqslant p<q.\end{cases} \tag{15}$$

From (14) and (15) we see that if $1\leqslant p<q$ and $p\leqslant 2$ then $c_1\left(\ell_q^n\left(\mathbb{F}_2^n,\|\cdot\|_p\right)\right)\asymp c_1\left(\ell_q^n(\ell_p^n)\right)$, while if $2<p<q$ then since $1/p-1/q<1-1/p-1/q$ we have $c_1\left(\ell_q^n\left(\mathbb{F}_2^n,\|\cdot\|_p\right)\right)=o\left(c_1\left(\ell_q^n(\ell_p^n)\right)\right)$.

The upper bound on $c_1(\ell_q^n\left(\mathbb{F}_2^n,\|\cdot\|_p\right))$ that appears in (14) will be proven in Section 4. We shall now show how the lower bound on $c_1(\ell_q^n\left(\mathbb{F}_2^n,\|\cdot\|_p\right))$ that appears in (14) quickly follows from Theorem 1.7. This will also establish Theorem 1.1 as a special case. So, suppose that $D\in[1,\infty)$ and $f:M_n(\mathbb{F}_2)\to L_1$ satisfies $\|x-y\|_{\ell_q^n(\ell_p^n)}\leqslant\|f(x)-f(y)\|_1\leqslant D\|x-y\|_{\ell_q^n(\ell_p^n)}$ for every $x,y\in M_n(\mathbb{F}_2)$. Our goal is to bound D from below. Since $\ell_q^n\left(\mathbb{F}_2^n,\|\cdot\|_p\right)$ contains an isometric copy of $\ell_q^{n-1}\left(\mathbb{F}_2^{n-1},\|\cdot\|_p\right)$, we may assume that n is even. By Theorem 1.7 applied to f we have

$$\frac1n\sum_{j=1}^n\sum_{x\in M_n(\mathbb{F}_2)}\left\|f\Big(x+\sum_{k=1}^n e_{jk}\Big)-f(x)\right\|_1\lesssim\frac{1}{n^n}\sum_{k\in\{1,\ldots,n\}^n}\sum_{x\in M_n(\mathbb{F}_2)}\left\|f\Big(x+\sum_{j=1}^n e_{jk_j}\Big)-f(x)\right\|_1. \tag{16}$$

But,

$$\sum_{j=1}^n\sum_{x\in M_n(\mathbb{F}_2)}\left\|f\Big(x+\sum_{k=1}^n e_{jk}\Big)-f(x)\right\|_1\geqslant 2^{n^2}\sum_{j=1}^n\Big\|\sum_{k=1}^n e_{jk}\Big\|_{\ell_q^n(\ell_p^n)}=2^{n^2}n^{1+\frac1p}, \tag{17}$$

and

$$\sum_{k\in\{1,\ldots,n\}^n}\sum_{x\in M_n(\mathbb{F}_2)}\Big\|f\Big(x+\sum_{j=1}^n e_{jk_j}\Big)-f(x)\Big\|_1\leqslant D2^{n^2}\sum_{k\in\{1,\ldots,n\}^n}\Big\|\sum_{j=1}^n e_{jk_j}\Big\|_{\ell_q^n(\ell_p^n)}=D2^{n^2}n^{n+\frac{1}{q}}. \quad (18)$$

For (16) to be compatible with (17) and (18) we must have $D\gtrsim n^{\frac{1}{p}-\frac{1}{q}}$. □

Remark 1.9. We only discussed L_1 embeddings of $\ell_q^n(\mathbb{F}_2^n,\|\cdot\|_p)$, but it is natural to also ask about embeddings of $\ell_q^m(\mathbb{F}_2^n,\|\cdot\|_p)$. However, it turns out that the case $m=n$ is the heart of the matter, i.e., the L_1 distortion of $\ell_q^m(\mathbb{F}_2^n,\|\cdot\|_p)$ is up to a constant factor the same as the L_1 distortion of $\ell_q^k\left(\mathbb{F}_2^k,\|\cdot\|_p\right)$ with $k=\min\{m,n\}$; see Remark 4.1 below.

2. Proof of Theorem 1.7

The stated sharpness of Theorem 1.7 is simple: consider the function $\varphi: M_n(\mathbb{F}_2)\to\mathbb{R}$ given by $\varphi(x)=(-1)^{x_{11}+\ldots+x_{nn}}$. For this choice of φ we have $\varphi(x+\sum_{k=1}^n e_{jk})=-\varphi(x)\in\{-1,1\}$ for every $x\in M_n(\mathbb{F}_2)$ and $j\in\{1,\ldots,n\}$. Consequently,

$$\sum_{j=1}^n\sum_{x\in M_n(\mathbb{F}_2)}\Big|\varphi\Big(x+\sum_{k=1}^n e_{jk}\Big)-\varphi(x)\Big|=n2^{n^2+1}. \quad (19)$$

Also, for every $x\in M_n(\mathbb{F}_2)$ and $k\in\{1,\ldots,n\}^n$ we have $\varphi(x+\sum_{j=1}^n e_{jk_j})=(-1)^{\ell(k)}\varphi(x)$, where $\ell(k)=|\{j\in\{1,\ldots,n\}:\ k_j=j\}|=\sum_{j=1}^n\mathbf{1}_{\{k_j=j\}}$. Consequently,

$$\begin{aligned}\sum_{k\in\{1,\ldots,n\}^n}\sum_{x\in M_n(\mathbb{F}_2)}\Big|\varphi\Big(x+\sum_{j=1}^n e_{jk_j}\Big)-\varphi(x)\Big|&=2^{n^2}\sum_{k\in\{1,\ldots,n\}^n}\Big(1-(-1)^{\sum_{j=1}^n\mathbf{1}_{\{k_j=j\}}}\Big)\\&=2^{n^2}n^n-2^{n^2}\prod_{j=1}^n\sum_{k=1}^n(-1)^{\mathbf{1}_{\{k=j\}}}=2^{n^2}\left(n^n-(n-2)^n\right). \quad (20)\end{aligned}$$

The identities (19) and (20) demonstrate that for every fixed $n\in\mathbb{N}$ the factor $2n/(n^n-(n-2)^n)$ in Theorem 1.7 cannot be replaced by any strictly smaller number.

Passing now to the proof of Theorem 1.7, we will actually prove the following statement, the case $p=1$ of which is Theorem 1.7 itself.

Theorem 2.1. *Suppose that $p\in(0,2]$ and $n\in 2\mathbb{N}$. Then for every $f: M_n(\mathbb{F}_2)\to L_p$ we have*

$$\sum_{j=1}^n\sum_{x\in M_n(\mathbb{F}_2)}\Big\|f\Big(x+\sum_{k=1}^n e_{jk}\Big)-f(x)\Big\|_p^p\leqslant\frac{2n}{n^n-(n-2)^n}\sum_{k\in\{1,\ldots,n\}^n}\sum_{x\in M_n(\mathbb{F}_2)}\Big\|f\Big(x+\sum_{j=1}^n e_{jk_j}\Big)-f(x)\Big\|_p^p.$$

Proof. By a classical theorem of Schoenberg [30], the metric space $(L_p,\|x-y\|_p^{p/2})$ admits an isometric embedding into L_2. Since the desired inequality is purely metric, i.e., it involves only distances between various values of f, it suffices to prove it for $p=2$ and then apply it to the composition of f with the Schoenberg isometry so as to deduce the desired inequality for general $p\in(0,2]$. In order to prove the case $p=2$, it suffices to prove the desired inequality when f is real-valued (deducing the case of L_2-valued f by integrating the resulting point-wise inequality).

Suppose then that $f: M_n(\mathbb{F}_2)\to\mathbb{R}$. We shall use below standard Fourier-analytic arguments on $M_n(\mathbb{F}_2)$, considered as a vector space (of dimension n^2) over $\mathbb{F}_2$. Specifically, one can write

$$f(x)=\sum_{A_1,\ldots,A_n\subseteq\{1,\ldots,n\}}\widehat{f}(A_1,\ldots,A_n)(-1)^{\sum_{j=1}^n\sum_{k\in A_j}x_{jk}},$$

where for every $A_1,\ldots,A_n\subseteq\{1,\ldots,n\}$,

$$\widehat{f}(A_1,\ldots,A_n)\stackrel{\mathrm{def}}{=}\frac{1}{2^{n^2}}\sum_{x\in M_n(\mathbb{F}_2)}(-1)^{\sum_{j=1}^n\sum_{k\in A_j}x_{jk}}f(x).$$

Then, for every $x\in M_n(\mathbb{F}_2)$ and $j\in\{1,\ldots,n\}$ we have

$$\begin{aligned}f\Big(x+\sum_{k=1}^n e_{jk}\Big)-f(x)&=\sum_{A_1,\ldots,A_n\subseteq\{1,\ldots,n\}}\widehat{f}(A_1,\ldots,A_n)\Big((-1)^{|A_j|}-1\Big)(-1)^{\sum_{s=1}^n\sum_{k\in A_s}x_{sk}}\\&=-2\sum_{\substack{A_1,\ldots,A_n\subseteq\{1,\ldots,n\}\\|A_j|\equiv 1\mod 2}}\widehat{f}(A_1,\ldots,A_n)(-1)^{\sum_{s=1}^n\sum_{k\in A_s}x_{sk}}.\end{aligned}$$

Hence, by the orthogonality of the functions $\{x\mapsto(-1)^{\sum_{s=1}^n\sum_{k\in A_s}x_{sk}}\}_{A_1,\ldots,A_n\subseteq\{1,\ldots,n\}}$ on $M_n(\mathbb{F}_2)$,

$$\begin{aligned}&\sum_{j=1}^n\sum_{x\in M_n(\mathbb{F}_2)}\Big(f\Big(x+\sum_{k=1}^n e_{jk}\Big)-f(x)\Big)^2\\&\qquad=2^{n^2+2}\sum_{A_1,\ldots,A_n\subseteq\{1,\ldots,n\}}|\{j\in\{1,\ldots,n\}:\ |A_j|\equiv 1\mod 2\}|\,\widehat{f}(A_1,\ldots,A_n)^2.\end{aligned}\tag{21}$$

At the same time, for every $x\in M_n(\mathbb{F}_2)$ and $k\in\{1,\ldots,n\}^n$ we have

$$f\Big(x+\sum_{j=1}^n e_{jk_j}\Big)-f(x)=\sum_{A_1,\ldots,A_n\subseteq\{1,\ldots,n\}}\widehat{f}(A_1,\ldots,A_n)\Big((-1)^{\sum_{j=1}^n\mathbf{1}_{A_j}(k_j)}-1\Big)(-1)^{\sum_{j=1}^n\sum_{k\in A_j}x_{jk}}.$$

Using orthogonality again, we therefore have

$$\begin{aligned}&\sum_{k\in\{1,\ldots,n\}^n}\sum_{x\in M_n(\mathbb{F}_2)}\Big(f\Big(x+\sum_{j=1}^n e_{jk_j}\Big)-f(x)\Big)^2\\&\qquad=2^{n^2}\sum_{k\in\{1,\ldots,n\}^n}\sum_{A_1,\ldots,A_n\subseteq\{1,\ldots,n\}}\widehat{f}(A_1,\ldots,A_n)^2\Big((-1)^{\sum_{j=1}^n\mathbf{1}_{A_j}(k_j)}-1\Big)^2\\&\qquad=2^{n^2+1}\sum_{A_1,\ldots,A_n\subseteq\{1,\ldots,n\}}\widehat{f}(A_1,\ldots,A_n)^2\sum_{k\in\{1,\ldots,n\}^n}\Big(1-(-1)^{\sum_{j=1}^n\mathbf{1}_{A_j}(k_j)}\Big).\end{aligned}\tag{22}$$

Fixing $A_1,\ldots,A_n\subseteq\{1,\ldots,n\}$, denote $S\stackrel{\mathrm{def}}{=}\{j\in\{1,\ldots,n\}:\ |A_j|\equiv 1\mod 2\}$. Since n is even, if $j\in S$ then $|A_j|\in\{1,\ldots,n-1\}$, and consequently $|2|A_j|-n|\leqslant n-2$. Hence,

$$\begin{aligned}\sum_{k\in\{1,\ldots,n\}^n}\Big(1-(-1)^{\sum_{j=1}^n\mathbf{1}_{A_j}(k_j)}\Big)&=n^n-\prod_{j=1}^n\sum_{k=1}^n(-1)^{\mathbf{1}_{A_j}(k)}\\&=n^n-\prod_{j=1}^n(n-2|A_j|)\geqslant n^n-\prod_{j=1}^n\big|2|A_j|-n\big|\geqslant n^n-n^{n-|S|}(n-2)^{|S|}.\end{aligned}\tag{23}$$

Since the mapping $|S|\mapsto\left(n^n-n^{n-|S|}(n-2)^{|S|}\right)/|S|$ is decreasing in $|S|$, it follows from (23) that

$$\sum_{k\in\{1,\ldots,n\}^n}\Big(1-(-1)^{\sum_{j=1}^n\mathbf{1}_{A_j}(k_j)}\Big)\geqslant\frac{n^n-(n-2)^n}{n}\,|\{j\in\{1,\ldots,n\}:\ |A_j|\equiv 1\mod 2\}|.\tag{24}$$

The desired inequality now follows by substituting (24) into (22) and recalling (21). $\square$

Remark 2.2. By the cut-cone decomposition of L_1 metrics (see e.g. [7]), the inequality of Theorem (1.7) is equivalent to the following (also sharp) isoperimetric-type inequality. For every and $n \in 2\mathbb{N}$ and $S \subseteq M_n(\mathbb{F}_2)$ we have

$$\sum_{j=1}^{n}\left|\left\{x \in S:\ x+\sum_{k=1}^{n} e_{jk} \notin S\right\}\right| \leqslant \frac{2n}{n^n-(n-2)^n}\sum_{k\in\{1,\ldots,n\}^n}\left|\left\{x \in S:\ x+\sum_{j=1}^{n} e_{jk_j} \notin S\right\}\right|. \quad (25)$$

Due to the simplicity of our proof of Theorem 1.7, we did not attempt to obtain a direct combinatorial proof of (25), though we believe that this should be doable (and potentially instructive). We also did not attempt to characterize the equality cases in (25).

Remark 2.3. In Remark 1.6 we have seen that (11) can hold true only if $n \in \mathbb{N}$ is even. However, this parity issue can remedied through the following (sharp) inequality, which holds true for every $n \in \mathbb{N}$, every $p \in (0,2]$ and every $f : M_n(\mathbb{F}_2) \to L_p$.

$$\begin{aligned}
&\frac{1}{n}\sum_{j=1}^{n}\sum_{x\in M_n(\mathbb{F}_2)}\sum_{y\in\mathbb{F}_2^n}\left\|f\Big(x+\sum_{k=1}^{n} y_j e_{jk}\Big)-f(x)\right\|_p^p\\
&\qquad\leqslant \frac{2}{1-\left(1-\frac{1}{n}\right)^n}\cdot\frac{1}{n^n 2^{n^2+n}}\sum_{k\in\{1,\ldots,n\}^n}\sum_{x\in M_n(\mathbb{F}_2)}\sum_{y\in\mathbb{F}_2^n}\left\|f\Big(x+\sum_{j=1}^{n} y_j e_{jk_j}\Big)-f(x)\right\|_p^p\\
&\qquad\leqslant \frac{2e}{e-1}\cdot\frac{1}{n^n 2^{n^2+n}}\sum_{k\in\{1,\ldots,n\}^n}\sum_{x\in M_n(\mathbb{F}_2)}\sum_{y\in\mathbb{F}_2^n}\left\|f\Big(x+\sum_{j=1}^{n} y_j e_{jk_j}\Big)-f(x)\right\|_p^p. \quad (26)
\end{aligned}$$

The distortion lower bounds that we obtained as a consequence of Theorem 1.7 also follow mutatis mutandis from (26), though they are weaker by a constant factor.

To prove (26), note that, exactly as in the beginning of the proof of Theorem 2.1, it suffices to prove (26) when $p = 2$ and $f : M_n(\mathbb{F}_2) \to \mathbb{R}$. Now, argue as in (22) to obtain the following identity.

$$\begin{aligned}
&\sum_{k\in\{1,\ldots,n\}^n}\sum_{x\in M_n(\mathbb{F}_2)}\sum_{y\in\mathbb{F}_2^n}\Big(f\Big(x+\sum_{j=1}^{n} y_j e_{jk_j}\Big)-f(x)\Big)^2\\
&\qquad= 2^{n^2+1}\sum_{A_1,\ldots,A_n\subseteq\{1,\ldots,n\}}\widehat{f}(A_1,\ldots,A_n)^2\sum_{k\in\{1,\ldots,n\}^n}\sum_{y\in\mathbb{F}_2^n}\Big(1-(-1)^{\sum_{j=1}^n y_j \mathbf{1}_{A_j}(k_j)}\Big). \quad (27)
\end{aligned}$$

For every $A_1,\ldots,A_n \subseteq \{1,\ldots,n\}$ and $k \in \{1,\ldots,n\}^n$ we have

$$\sum_{y\in\mathbb{F}_2^n}\Big(1-(-1)^{\sum_{j=1}^n y_j \mathbf{1}_{A_j}(k_j)}\Big) = \begin{cases} 2^n & \text{if } k_j \in A_j \text{ for some } j \in \{1,\ldots,n\},\\ 0 & \text{otherwise.}\end{cases}$$

Hence, denoting $T \stackrel{\mathrm{def}}{=} \{j \in \{1,\ldots,n\}:\ A_j \neq \emptyset\} \supseteq \{j \in \{1,\ldots,n\}:\ |A_j| \equiv 1 \mod 2\} \stackrel{\mathrm{def}}{=} S$,

$$\begin{aligned}
&\sum_{k\in\{1,\ldots,n\}^n}\sum_{y\in\mathbb{F}_2^n}\Big(1-(-1)^{\sum_{j=1}^n y_j \mathbf{1}_{A_j}(k_j)}\Big) = 2^n\sum_{k\in\{1,\ldots,n\}^n}\Big(1-\mathbf{1}_{\{\forall\, j\in\{1,\ldots,n\},\quad k_j\notin A_j\}}\Big)\\
&\qquad= 2^n\Big(n^n-\prod_{j=1}^{n}(n-|A_j|)^n\Big) \geqslant 2^n\Big(n^n-n^{n-|T|}(n-1)^{|T|}\Big) \geqslant 2^n\left(n^n-(n-1)^n\right)\frac{|T|}{n}. \quad (28)
\end{aligned}$$

Consequently, by (27) and (28), combined with the fact that $|T| \geqslant |S|$, we have

$$\sum_{k\in\{1,\ldots,n\}^n}\sum_{x\in M_n(\mathbb{F}_2)}\sum_{y\in\mathbb{F}_2^n}\Big(f\Big(x+\sum_{j=1}^n y_je_{jk_j}\Big)-f(x)\Big)^2$$

$$\geqslant \frac{2^{n^2+n+1}\left(n^n-(n-1)^n\right)}{n}\sum_{A_1,\ldots,A_n\subseteq\{1,\ldots,n\}}|\{j\in\{1,\ldots,n\}:\ |A_j|\equiv 1 \mod 2\}|\,\widehat{f}(A_1,\ldots,A_n)^2$$

$$\stackrel{(21)}{=}\frac{2^n\left(n^n-(n-1)^n\right)}{2n}\sum_{j=1}^n\sum_{x\in M_n(\mathbb{F}_2)}\Big(f\Big(x+\sum_{k=1}^n e_{jk}\Big)-f(x)\Big)^2.$$

This completes the proof of (26).

Remark 2.4. As stated in the Introduction, the "vanilla" metric Kwapień–Schütt inequality (9) cannot hold true in any non-singleton metric space (X,d_X). To see this, note first that we have already seen in Remark 1.6 that if $n\in\mathbb{N}$ is odd then (9) fails to hold true for any $K>0$. So, suppose that $n\geqslant 4$ is an even integer. It suffice to deal with $X=\{-1,1\}\subseteq\mathbb{R}$. Define $\psi: M_n(\mathbb{F}_2)\to\{-1,1\}$ by $\psi(x)=(-1)^{x_{11}+\sum_{j=2}^n\sum_{k=4}^n x_{jk}}$. For every $x\in M_n(\mathbb{F}_2)$ we have $\psi(x+\sum_{k=1}^n e_{1k})=-\psi(x)$, and for $j\in\{2,\ldots,n\}$ we have $\psi(x+\sum_{k=1}^n e_{jk})=(-1)^{n-3}\psi(x)=-\psi(x)$, since n is even. Consequently,

$$\sum_{j=1}^n\sum_{x\in M_n(\mathbb{F}_2)}\Big|\psi\Big(x+\sum_{k=1}^n e_{jk}\Big)-\psi(x)\Big|=n2^{n^2+1}. \tag{29}$$

At the same time, for every $x\in M_n(\mathbb{F}_2)$ and $\pi\in S_n$ we have

$$\psi\Big(x+\sum_{j=1}^n e_{j\pi(j)}\Big)=(-1)^{\mathbf{1}_{\{\pi(1)=1\}}+\sum_{j=2}^n\mathbf{1}_{\{\pi(j)\geqslant 4\}}}\psi(x). \tag{30}$$

If $\pi(1)=1$ then $\{2,3\}\subseteq\{\pi(2),\ldots,\pi(n)\}$ and therefore $\mathbf{1}_{\{\pi(1)=1\}}+\sum_{j=2}^n\mathbf{1}_{\{\pi(j)\geqslant 4\}}=n-2$ is even. If $\pi(1)\in\{1,\ldots,n\}\smallsetminus\{1,2,3\}$ then $\{\pi(2),\ldots,\pi(n)\}\supseteq\{1,2,3\}$ and consequently we have that $\mathbf{1}_{\{\pi(1)=1\}}+\sum_{j=2}^n\mathbf{1}_{\{\pi(j)\geqslant 4\}}=n-4$ is even. In the remaining case $\pi(1)\in\{2,3\}$ we have that $\mathbf{1}_{\{\pi(1)=1\}}+\sum_{j=2}^n\mathbf{1}_{\{\pi(j)\geqslant 4\}}=n-3$ is odd. Hence, by (30) we have

$$\sum_{\pi\in S_n}\sum_{x\in M_n(\mathbb{F}_2)}\Big|\psi\Big(x+\sum_{j=1}^n e_{j\pi(j)}\Big)-\psi(x)\Big|=2^{n^2+1}\left|\{\pi\in S_n:\ \pi(1)\in\{2,3\}\}\right|=2^{n^2+2}(n-1)!. \tag{31}$$

By contrasting (29) with (31) we see that if (9) holds true then necessarily $K\geqslant n/2$.

3. Uniform and coarse nonembeddability

A metric space (X,d_X) is said to admit a uniform embedding into a Banach space $(Z,\|\cdot\|_Z)$ if there exists an injective mapping $f:X\to Z$ and nondecreasing functions $\alpha,\beta:(0,\infty)\to(0,\infty)$ with $\lim_{t\to 0}\beta(t)=0$ such that $\alpha(d_X(a,b))\leqslant\|f(a)-f(b)\|_Z\leqslant\beta(d_X(a,b))$ for all distinct $a,b\in X$. Similarly, (X,d_X) is said to admit a coarse embedding into a Banach space $(Z,\|\cdot\|_Z)$ if there exists an injective mapping $f:X\to Z$ and nondecreasing functions $\alpha,\beta:(0,\infty)\to(0,\infty)$ with $\lim_{t\to\infty}\alpha(t)=\infty$ for which $\alpha(d_X(a,b))\leqslant\|f(a)-f(b)\|_Z\leqslant\beta(d_X(a,b))$ for all distinct $a,b\in X$.

The space $\ell_2(\ell_1)$ does not admit a uniform or coarse embedding into L_1. Indeed, by [1] in the case of uniform embeddings and by [26] in the case of coarse embeddings, this would imply that $\ell_2(\ell_1)$ is linearly isomorphic to a subspace of L_0, which is proved to be impossible in [14, Theorem 4.2].

Theorem 1.7 yields a new proof that $\ell_2(\ell_1)$ does not admit a uniform or coarse embedding into L_1. Indeed, suppose that $\alpha,\beta:(0,\infty)\to(0,\infty)$ are nondecreasing and $f:\ell_2(\ell_1)\to L_1$ satisfies

$$\forall\, x,y\in\ell_2(\ell_1),\qquad \alpha\left(\|f(x)-f(y)\|_{\ell_2(\ell_1)}\right)\leqslant\|f(x)-f(y)\|_1\leqslant\beta\left(\|f(x)-f(y)\|_{\ell_2(\ell_1)}\right). \tag{32}$$

For every $s \in (0,\infty)$ and $n \in 2\mathbb{N}$, apply Theorem 1.7 to the mapping $f_s : M_n(\mathbb{F}_2) \to L_1$ given by $f_s(x) = f(sx)$. The resulting inequality, when combined with (32), implies that $\alpha(sn) \lesssim \beta(s\sqrt{n})$. Choosing $s = 1/\sqrt{n}$ shows that $\alpha(\sqrt{n}) \lesssim \beta(1)$, so f is not a coarse embedding, and choosing $s = 1/n$ shows that $\beta(1/\sqrt{n}) \gtrsim \alpha(1) > 0$, so f is not a uniform embedding.

Observe that since, by [13], L_p is isometric to a subset of L_1 when $p \in [1,2]$, the above discussion implies that $\ell_2(\ell_1)$ does not admit a uniform or coarse embedding into L_p for every $p \in [1,2]$. Passing now to an examination of the uniform and coarse embeddability of $\ell_2(\ell_1)$ into L_p for $p > 2$, observe first that since, by [1] and [12], when $p > 2$ there is no uniform or coarse embedding of L_p into L_1, the fact that $\ell_2(\ell_1)$ does not admit a uniform or coarse embedding into L_1 does not imply that $\ell_2(\ell_1)$ fails to admit such an embedding into L_p. An inspection of the above argument reveals that in order to show that $\ell_2(\ell_1)$ does not admit a uniform or coarse embedding into L_p it would suffice to establish the following variant of Theorem 2.1 when $p > 2$: there exits $C_p, \theta_p \in (0,\infty)$ such that for every $n \in 2\mathbb{N}$, every $f : M_n(\mathbb{F}_2) \to L_p$ satisfies

$$\frac{1}{n}\sum_{j=1}^{n}\sum_{x\in M_n(\mathbb{F}_2)}\Big\|f\Big(x+\sum_{k=1}^{n}e_{jk}\Big)-f(x)\Big\|_p^{\theta_p} \leqslant \frac{C_p}{n^n}\sum_{k\in\{1,\ldots,n\}^n}\sum_{x\in M_n(\mathbb{F}_2)}\Big\|f\Big(x+\sum_{j=1}^{n}e_{jk_j}\Big)-f(x)\Big\|_p^{\theta_p}. \quad (33)$$

However, no such extension of Theorem 2.1 to the range $p > 2$ is possible. Indeed, since ℓ_2 is linearly isometric to a subspace of L_p, we may fix a linear isometry $U : \ell_p^n(\ell_2^n) \to L_p$. Define $f : M_n(\mathbb{F}_2) \to L_p$ by $f(x) = \sum_{j=1}^n \sum_{k=1}^n x_{jk} U(e_{jk})$. For this choice of f, (33) becomes

$$2^{n^2} n^{\frac{\theta_p}{2}} = \frac{1}{n}\sum_{j=1}^{n}\sum_{x\in M_n(\mathbb{F}_2)}\Big\|\sum_{k=1}^{n}e_{jk}\Big\|_{\ell_p^n(\ell_2^n)}^{\theta_p} \overset{(33)}{\leqslant} \frac{C_p}{n^n}\sum_{k\in\{1,\ldots,n\}^n}\sum_{x\in M_n(\mathbb{F}_2)}\Big\|\sum_{j=1}^{n}e_{jk_j}\Big\|_{\ell_p^n(\ell_2^n)}^{\theta_p} = 2^{n^2} C_p n^{\frac{\theta_p}{p}},$$

which is a contradiction for large enough $n \in 2\mathbb{N}$, because $p > 2$. Thus, it was crucial to assume in Theorem 2.1 that $p \leqslant 2$. When $p \geqslant 4$, this is accentuated by the following proposition.

Proposition 3.1. *For every $p \geqslant 4$ the exists a mapping $F_p : \ell_2(\ell_1) \to L_p$ that satisfies*

$$\forall\, x, y \in \ell_2(\ell_1), \qquad \|F_p(x) - F_p(y)\|_p = \|x-y\|_{\ell_2(\ell_1)}^{\frac{2}{p}}. \quad (34)$$

Thus $\ell_2(\ell_1)$ admits an embedding into L_p that is both uniform and coarse.

Proof. Fix $T : \ell_1 \to L_2$ such that

$$\forall\, x, y \in \ell_1, \qquad \|T(x) - T(y)\|_2 = \sqrt{\|x-y\|_1}. \quad (35)$$

See [7] for the existence of such T (an explicit formula for T appear in [23, Section 3]). By a classical theorem of Schoenberg [30], since $4/p \leqslant 1$ there exists a mapping $\sigma_p : L_2 \to L_2$ that satisfies

$$\forall\, x, y \in L_2 \qquad \|\sigma_p(x) - \sigma_p(y)\|_2 = \|x-y\|_2^{\frac{4}{p}}. \quad (36)$$

Fix also an isometric embedding $S : L_2 \to L_p$ and define $F_p : \ell_2(\ell_1) \to \ell_p(L_p) \cong L_p$ by

$$\forall\, x \in \ell_2(\ell_1), \qquad F_p(x) \overset{\mathrm{def}}{=} \left(S \circ \sigma_p \circ T(x_j)\right)_{j=1}^{\infty}.$$

Then, for every $x, y \in \ell_2(\ell_1)$ we have

$$\|F_p(x) - F_p(y)\|_{\ell_p(L_p)} = \Big(\sum_{j=1}^{\infty}\|S(\sigma_p(T(x_j))) - S(\sigma_p(T(y_j)))\|_p^p\Big)^{\frac{1}{p}}$$

$$= \Big(\sum_{j=1}^{\infty}\|\sigma_p(T(x_j)) - \sigma_p(T(y_j))\|_2^p\Big)^{\frac{1}{p}} \overset{(36)}{=} \Big(\sum_{j=1}^{\infty}\|T(x_j) - T(y_j)\|_2^4\Big)^{\frac{1}{p}} \overset{(35)}{=} \Big(\sum_{j=1}^{\infty}\|x_j - y_j\|_1^2\Big)^{\frac{1}{p}},$$

which is precisely the desired requirement (34). $\square$

The above proof of Proposition 3.1 used the fact that $p \geqslant 4$ in order for (36) to hold true. It is therefore natural to ask Question 3.2 below. Analogous questions could be asked for uniform and coarse embeddings of $\ell_{p_1}(\ell_{p_2})$ into L_{p_3} (or even into $\ell_{p_3}(L_{p_4})$), and various partial results could be obtained using similar arguments (at times with the embedding in (36) replaced by the embedding of [21, Remark 5.10]). We shall not pursue this direction here because it yields incomplete results.

Question 3.2. Suppose that $2 < p < 4$. Does $\ell_2(\ell_1)$ admit a uniform or coarse embedding into L_p?

Remark 3.3. In the Introduction we asked whether or not the Schatten trace class S_1 is a KS metric space. The approach of Section 2 seems inherently insufficient to address this question. Indeed, we treated L_1 by relating its metric to Hilbert space through the isometric embedding of $(L_1, \sqrt{\|x-y\|_1})$, while S_1 is not even uniformly homeomorphic to a subset of Hilbert space (this follows from [1] combined with e.g. [14] and the classical linear nonembeddability result of [20]). For this reason we believe that asking about the validity of (11) in S_1 is worthwhile beyond its intrinsic interest, as a potential step towards addressing more general situations in which one cannot reduce the question to (nonlinear) Hilbertian considerations.

4. Embeddings

In this section we shall justify the remaining (simple) embedding statements that were given without proof in the Introduction, starting with the proof of Lemma 1.3.

Proof of Lemma 1.3. Recall that we are given $n \in \mathbb{N}$, finite subsets $X_1, \ldots, X_n \subseteq \ell_1$, and we denote $S = X_1 \times \ldots \times X_n \subseteq \ell_2^n(\ell_1)$. Thus $|S| = \prod_{j=1}^n |X_i|$. We may assume without loss of generality that $|X_j| > 1$ for all $j \in \{1, \ldots, n\}$. Write

$$J \stackrel{\mathrm{def}}{=} \left\{ j \in \{1, \ldots, n\} : \ |X_j| > \exp\left(\frac{\sqrt{\log |S|}}{\log\log |S|} \right) \right\}. \tag{37}$$

Then

$$|S| \geqslant \prod_{j\in J} |X_j| > \exp\left(\frac{|J|\sqrt{\log |S|}}{\log\log |S|} \right) \implies |J| < \sqrt{\log |S|}\log\log |S|. \tag{38}$$

By the main result of [2], for every $j \in \{1, \ldots, n\}$ there exists $f_j : X_j \to \ell_2$ such that

$$\forall\, u, v \in X_j, \qquad \|u - v\|_1 \leqslant \|f_j(u) - f_j(v)\|_2 \lesssim \sqrt{\log |X_j|}\log\log |X_j| \cdot \|u - v\|_1. \tag{39}$$

We shall fix from now on an isometric embedding $T : \ell_2^{\{1,\ldots,n\}\smallsetminus J}(\ell_2) \to L_1$.

Define $\phi : S \to (\ell_1^J(\ell_1) \oplus L_1)_1$, where $(\ell_1^J(\ell_1) \oplus L_1)_1$ is the corresponding ℓ_1-direct sum, by setting

$$\phi(u) \stackrel{\mathrm{def}}{=} ((u_j)_{j\in J}) \oplus T\left((f_j(u_j))_{j \in \{1,\ldots,n\}\smallsetminus J} \right).$$

Then for every $u, v \in S$ we have

$$\begin{aligned} \|\phi(u) - \phi(v)\|_{(\ell_1^J(\ell_1)\oplus L_1)_1} &= \sum_{j\in J} \|u_j - v_j\|_1 + \Bigg(\sum_{j\in\{1,\ldots,n\}\smallsetminus J} \|f_j(u_j) - f_j(v_j)\|_2^2 \Bigg)^{\frac12} \\ &\geqslant \Bigg(\sum_{j\in J} \|u_j - v_j\|_1^2 \Bigg)^{\frac12} + \Bigg(\sum_{j\in\{1,\ldots,n\}\smallsetminus J} \|u_j - v_j\|_1^2 \Bigg)^{\frac12} \geqslant \|u - v\|_{\ell_2^n(\ell_1)}, \end{aligned} \tag{40}$$

where in the first inequality of (40) we used the leftmost inequality in (39). The corresponding upper bound is deduced as follows from Cauchy–Schwarz, the rightmost inequality in (39), the

definition of J in (37), and the upper bound on $|J|$ in (38).

$$
\begin{aligned}
&\|\phi(u)-\phi(v)\|_{(\ell_1^J(\ell_1)\oplus L_1)_1}\\
&=\sum_{j\in J}\|u_j-v_j\|_1+\Bigg(\sum_{j\in\{1,\ldots,n\}\smallsetminus J}\|f_j(u_j)-f_j(v_j)\|_2^2\Bigg)^{\frac12}\\
&\lesssim\sqrt{|J|}\Bigg(\sum_{j\in J}\|u_j-v_j\|_1^2\Bigg)^{\frac12}+\Bigg(\max_{j\in\{1,\ldots,n\}\smallsetminus J}\sqrt{\log|X_j|}\log\log|X_j|\Bigg)\Bigg(\sum_{j\in\{1,\ldots,n\}\smallsetminus J}\|u_j-v_j\|_1^2\Bigg)^{\frac12}\\
&\leqslant\sqrt[4]{\log|S|}\sqrt{\log\log|S|}\Bigg(\sum_{j\in J}\|u_j-v_j\|_1^2\Bigg)^{\frac12}+\frac{\sqrt[4]{\log|S|}\log\left(\frac{\sqrt{\log|S|}}{\log\log|S|}\right)}{\sqrt{\log\log|S|}}\Bigg(\sum_{j\in\{1,\ldots,n\}\smallsetminus J}\|u_j-v_j\|_1^2\Bigg)^{\frac12}\\
&\lesssim\sqrt[4]{\log|S|}\sqrt{\log\log|S|}\cdot\|u-v\|_{\ell_2^n(\ell_1)}. \qquad\square
\end{aligned}
$$

We shall next justify the upper bound on $c_1(\ell_q^n(\mathbb{F}_2^n,\|\cdot\|_p))$ in (14). Recall that we are assuming here that $q>p\geqslant 1$. Since for every $x,y\in M_n(\mathbb{F}_2)$ we have

$$
\|x-y\|_{\ell_q^n(\ell_p^n)}=\Bigg(\sum_{j=1}^n\|x_j-y_j\|_p^q\Bigg)^{\frac1q}=\Bigg(\sum_{j=1}^n\|x_j-y_j\|_1^{\frac qp}\Bigg)^{\frac1q},
$$

by Hölder's inequality

$$
\frac{\|x-y\|_{\ell_1^n(\ell_1^n)}^{\frac1p}}{n^{\frac1p-\frac1q}}\leqslant\|x-y\|_{\ell_q^n(\ell_p^n)}\leqslant\|x-y\|_{\ell_1^n(\ell_1^n)}^{\frac1p}. \tag{41}
$$

By a classical theorem of Bretagnolle, Dacunha-Castelle and Krivine [6] (see also [31, Theorem 5.11]), for every $\alpha\in(0,1]$ the metric space $(L_1,\|x-y\|_1^\alpha)$ admits an isometric embedding into L_1. Hence, the metric space

$$
\left(M_n(\mathbb{F}_2),\|x-y\|_{\ell_1^n(\ell_1^n)}^{\frac1p}\right)
$$

admits an isometric embedding into L_1, and consequently (41) implies that

$$
c_1\left(\ell_q^n(\mathbb{F}_2^n,\|\cdot\|_p)\right)\leqslant n^{\frac1p-\frac1q}. \qquad\square
$$

Remark 4.1. Arguing similarly to the above discussion also justifies the assertion in Remark 1.9. Indeed, suppose that $m,n\in\mathbb{N}$ and $q>p\geqslant 1$. Then by Hölder's inequality for every $x,y\in\ell_q^m(\mathbb{F}_2^n)$,

$$
\frac{\|x-y\|_{\ell_1^m(\ell_1^n)}^{\frac1p}}{m^{\frac1p-\frac1q}}=\frac{\|x-y\|_{\ell_p^m(\ell_p^n)}}{m^{\frac1p-\frac1q}}\leqslant\|x-y\|_{\ell_q^m(\ell_p^n)}\leqslant\|x-y\|_{\ell_p^m(\ell_p^n)}=\|x-y\|_{\ell_1^m(\ell_1^n)}^{\frac1p}. \tag{42}
$$

Also, because for every $x,y\in\ell_q^m(\mathbb{F}_2^n)$ and $j\in\{1,\ldots,n\}$ we have $\|x_j-y_j\|_1\in\{0,\ldots,n\}$,

$$
\|x-y\|_{\ell_q^m(\ell_p^n)}=\Bigg(\sum_{j=1}^n\|x_j-y_j\|_1^{\frac qp}\Bigg)^{\frac1q}\in\left[\|x-y\|_{\ell_1^m(\ell_1^n)}^{\frac1q},n^{\frac1p-\frac1q}\|x-y\|_{\ell_1^m(\ell_1^n)}^{\frac1q}\right]. \tag{43}
$$

Since the metric spaces

$$
\left(M_n(\mathbb{F}_2),\|x-y\|_{\ell_1^n(\ell_1^n)}^{\frac1p}\right)\qquad\text{and}\qquad\left(M_n(\mathbb{F}_2),\|x-y\|_{\ell_1^n(\ell_1^n)}^{\frac1q}\right)
$$

admit an isometric embedding into L_1, it follows from (42) and (43) that

$$c_1\left(\ell_q^m\left(\mathbb{F}_2^n, \|\cdot\|_p\right)\right) \leqslant \min\left\{m^{\frac{1}{p}-\frac{1}{q}}, n^{\frac{1}{p}-\frac{1}{q}}\right\}. \tag{44}$$

Since $\ell_q^m\left(\mathbb{F}_2^n, \|\cdot\|_p\right)$ contains an isometric copy of $\ell_q^{\min\{m,n\}}\left(\mathbb{F}_2^{\min\{m,n\}}, \|\cdot\|_p\right)$, by (14) and (44),

$$c_1\left(\ell_q^m\left(\mathbb{F}_2^n, \|\cdot\|_p\right)\right) \asymp c_1\left(\ell_q^{\min\{m,n\}}\left(\mathbb{F}_2^{\min\{m,n\}}, \|\cdot\|_p\right)\right),$$

as required.

We end with a brief justification of (15). If $1 \leqslant p < q$ and $p \leqslant 2$ then $c_1\left(\ell_q^n(\ell_p^n)\right) \gtrsim n^{1/p-1/q}$, as proved by Kwapień and Schütt [16]. The reverse inequality follows from the fact that the $\ell_q^n(\ell_p^n)$ norm is $n^{1/p-1/q}$-equivalent to the $\ell_p^n(\ell_p^n)$ norm, and from the fact [13] that ℓ_p is isometric to a subspace of L_1 when $p \leqslant 2$. When $q > p > 2$, the ℓ_p^n norm is $n^{1/2-1/q}$-equivalent to the ℓ_2^n norm and the ℓ_q^n norm is $n^{1/2-1/q}$-equivalent to the ℓ_2^n norm. So, the $\ell_q^n(\ell_p^n)$ norm is $n^{1-1/p-1/q}$-equivalent to the $\ell_2^n(\ell_2^n)$ norm, which embeds isometrically into L_1. For the matching lower bound, suppose that $T : \ell_q^n(\ell_p^n) \to L_1$ is an injective linear mapping. Since L_1 has cotype 2 (see e.g. [19]),

$$\begin{aligned}\frac{n^2}{\|T^{-1}\|^2} \leqslant \sum_{j=1}^n\sum_{k=1}^n \|Te_{jk}\|_1^2 &\lesssim \frac{1}{2^{n^2}} \sum_{\varepsilon\in\{-1,1\}^{n^2}} \Big\|\sum_{j=1}^n\sum_{k=1}^n \varepsilon_{jk}Te_{jk}\Big\|_1^2 \\ &\leqslant \frac{\|T\|^2}{2^{n^2}} \sum_{\varepsilon\in\{-1,1\}^{n^2}} \Big\|\sum_{j=1}^n\sum_{k=1}^n e_{jk}\Big\|_{\ell_q^n(\ell_p^n)}^2 = \|T\|^2\cdot n^{\frac{2}{p}+\frac{2}{q}}. \end{aligned} \tag{45}$$

By (45) we have $\|T\|\cdot\|T^{-1}\| \gtrsim n^{1-1/p-1/q}$. The fact that $c_1\left(\ell_q^n(\ell_p^n)\right) \gtrsim n^{1-1/p-1/q}$ now follows by a standard differentiation argument; see e.g. [4, Chapter 7] (alternatively, one could repeat the above argument mutatis mutandis, while using the fact that L_1 has metric cotype 2 directly; see [22]).

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